Mathematical Physics B.Sc (Physics) – Part-III

03/06/20

* Laplacian in Orthogonal Curvilinear Coordinate:-

We have already discussed gradient in orthogonal curvilinear Coordinates ~~are g~~ is given by

$$\nabla\psi = \frac{\hat{e}_1}{h_1}\frac{\partial\psi}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial\psi}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\psi}{\partial u_3} \quad \text{—— ①}$$

Laplacian is defined as — $\nabla^2\psi$ {for the function ψ}

$$\nabla^2\psi = \nabla\cdot\nabla\psi$$

$$= \nabla\cdot\left[\frac{\hat{e}_1}{h_1}\frac{\partial\psi}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial\psi}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\psi}{\partial u_3}\right] \quad \text{—— ②}$$

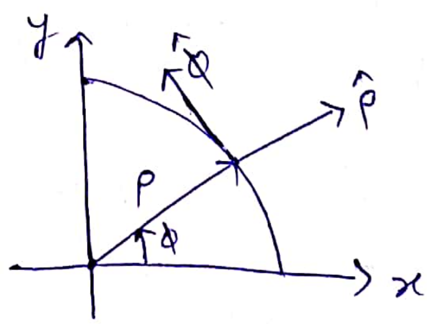
Since $\nabla\cdot\vec{V}$ for $\vec{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$ is given by

$$\nabla\cdot\vec{V} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3V_1) + \frac{\partial}{\partial u_2}(h_1h_3V_2) + \frac{\partial}{\partial u_3}(h_1h_2V_3)\right] \quad \text{—— ③}$$

$$\boxed{\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]} \quad \text{—— ④}$$

* Plane Polar Coordinates:

Curvilinear coordinates (u_1, u_2) are relabeled by (ρ, ϕ), $\left\{\begin{matrix} 0 \le \rho < \infty \\ 0 \le \phi \le 2\pi \end{matrix}\right\}$



Coordinate Surfaces:

1. $\rho = \sqrt{x^2+y^2} = \text{const.}$
2. $\phi = \tan^{-1}(y/x) = \text{const.}$

Transformation equations: $\boxed{x = \rho\cos\phi, \quad y = \rho\sin\phi}$

* Right Circular Cylindrical Coordinates:

Curvilinear coordinates (u_1, u_2, u_3) are relabeled by (ρ, ϕ, z).

$\rho \rightarrow$ Perpendicular distance from the Z-axis.

Limits of ρ, ϕ, z are —

$0 \leq \rho < \infty$, $0 \leq \phi \leq 2\pi$, and $-\infty < z < \infty$.

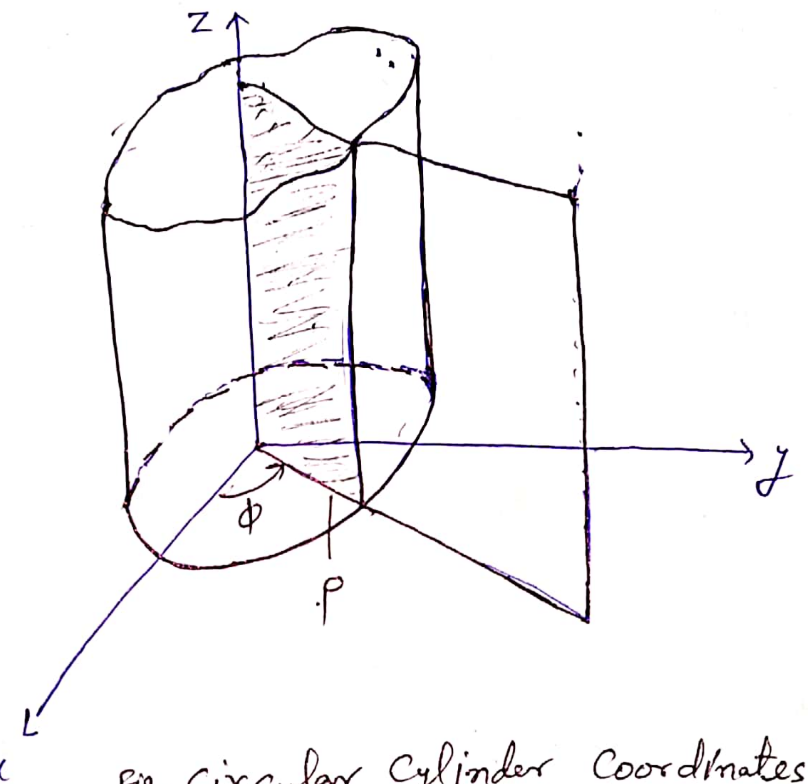


Fig- Circular Cylinder Coordinates.

Coordinate surfaces:

1. z axis as a common axis $\quad \rho = \sqrt{x^2+y^2} = \text{const.}$

2. Half-plane through z-axis. $\quad \phi = \tan^{-1}\left(\frac{y}{x}\right) = \text{const.}$

3. Planes parallel to xy-plane.

$$z = \text{constant.}$$

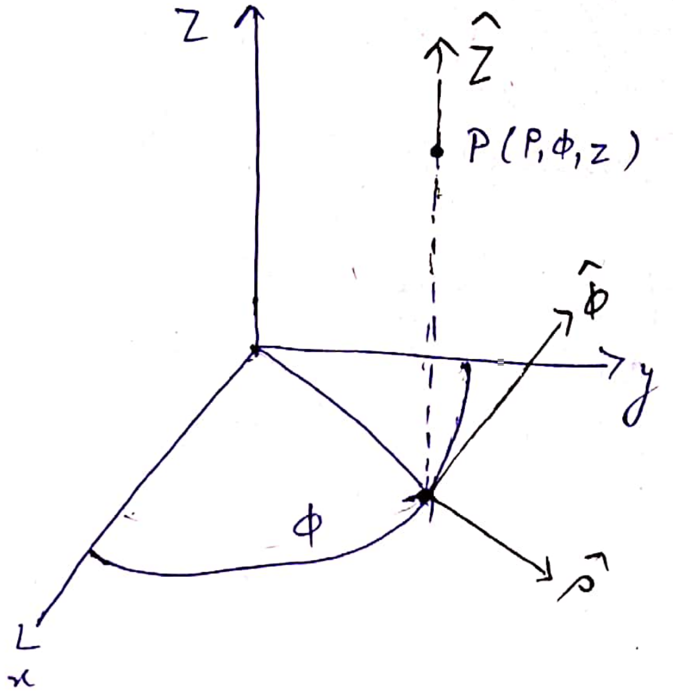


Fig showing unit vectors.

Transformation relations—

$$\boxed{x = \rho\cos\phi\,,\; y = \rho\sin\phi\,,\; z = z}$$

* Spherical Polar Coordinates :

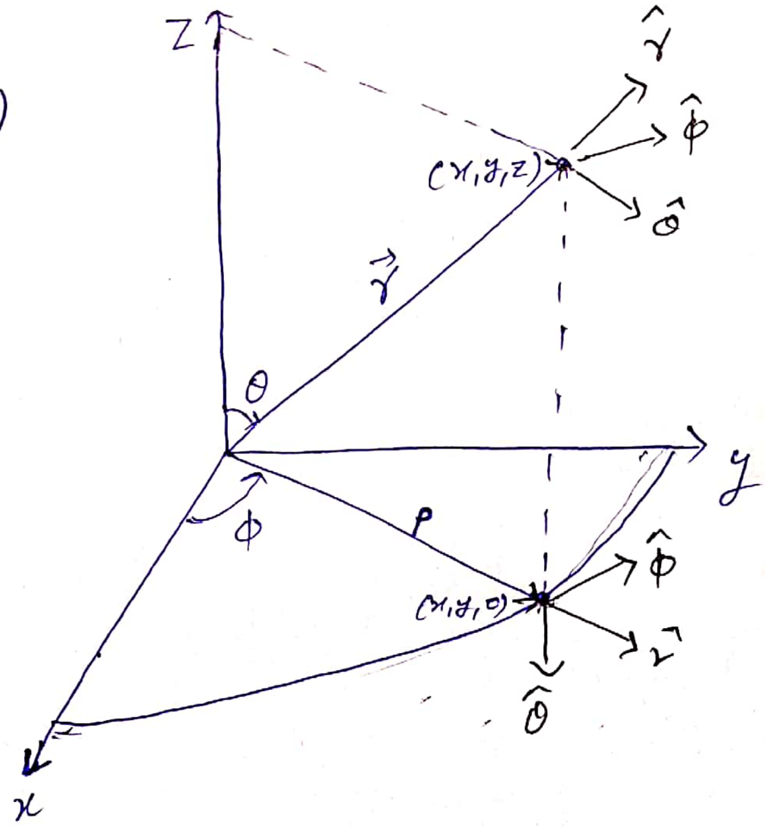
$(u_1, u_2, u_3) \longrightarrow (r, \theta, \phi)$

limits of r, θ, ϕ

$0 \le r < \infty$,

$0 \le \theta \le \pi$,

$0 \le \phi \le 2\pi$



coordinate Surfaces,

1. $r = \sqrt{(x^2+y^2+z^2)} = \text{const.}$

2. $\theta = \arccos \dfrac{z}{\sqrt{x^2+y^2+z^2}} = \text{const.}$

3. $\phi = \tan^{-1}\left(\dfrac{y}{x}\right) = \text{const}$

Transformations equations:

$$\boxed{x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.}$$